Calculus 1, Workshop 7: Optimization

Dogs Know Calculus?!

PAULA DREWNIANY, SUE MCGARRY, JEN TYNE

References:

 Tim Penning's Article First Appeared in:

Pennings, T.J. 2003. Do dogs know calculus? *College Mathematics Journal* 34(May):178-182. Available at: http://www.maa.org/features/elvisdog.pdf.

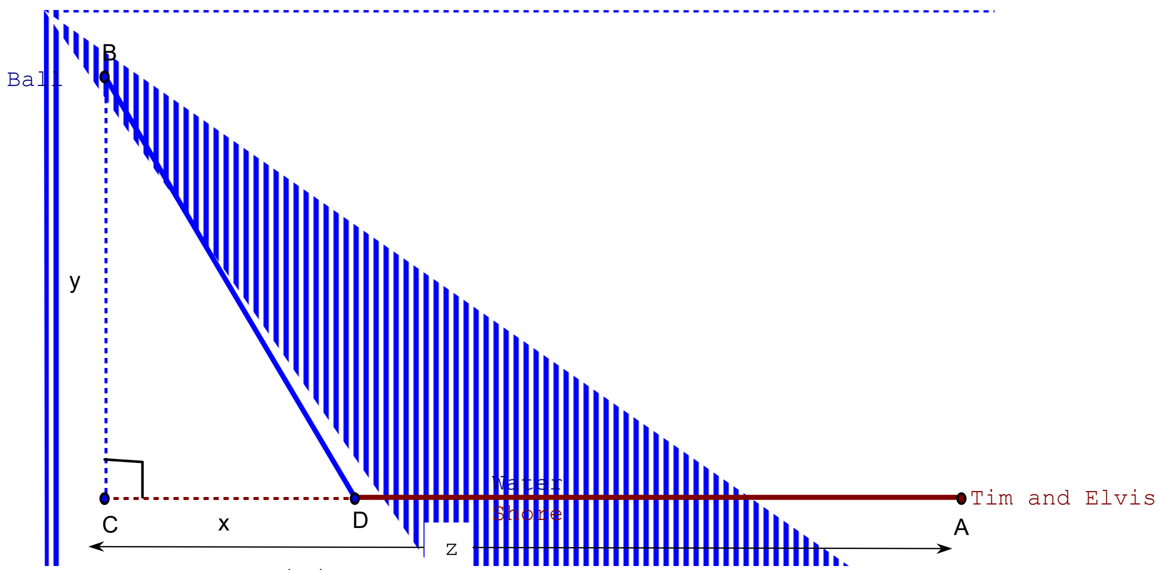
Peterson, Ivars, 2003, A Dog, a Ball, and Calculus, *Science News,24*(June 14)
Available at http://sciencenews.org/20030614/mathtrek.asp or http://www.maa.org/mathland/mathtrek%5F06%5F09%5F03.html

Stewart, Calculus,2nd ed, 2001, Brooks Cole, CA, Section 4.6, pg. 311.

In his article, “Do Dogs Know Calculus?”, Tim Pennings relates how his dog, Elvis, whose running speed differs from his swimming speed, consistently enters the water at the location that minimizes the time it takes him to retrieve a ball thrown into the water from the shore. In today’s workshop, we will first attempt to demonstrate that we, too, could determine that optimum location to enter the water. Second, using the data that Tim Pennings supplies, we will try to verify that Elvis really does know Calculus.

The Problem

Elvis and Tim are standing on the shore, at point A, and Tim throws the ball into the water, to point B. Elvis runs down the shore to point D, and jumps into the water. He swims to point B, and retrieves the ball. We wish to determine the location of point D that results in the minimum time for Elvis to reach the ball.



- Line, $\overleftrightarrow{AC}$, represents the shore line.
- Point D is on line $\overleftrightarrow{AC}$
- Line, $\overleftrightarrow{BC}$, is perpendicular to $\overleftrightarrow{AC}$.
- Let y be the distance from point C to point B.
- Let x be the distance from point C to point D.
- Let z be the distance from point C to point A.
- Let r represent Elvis' running speed
- Let s represent Elvis' swimming speed

Part I: Preliminary Questions

In this problem, we wish to minimize the time it takes Elvis to reach the ball, and we note that the time depends on the point at which Elvis enters the water. Answer the following questions qualitatively, thinking only of the physical situation and not about the calculus.

For questions 1, 2, 3, and 4 decide where, or approximately where, Elvis should enter the water in order to minimize his time to the ball. Justify your answer with a physical argument:

1. If Elvis swims much faster than he runs, at what point should he enter the water?

2. If Elvis swims at the same rate that he runs, where should he enter the water?

3. If Elvis runs a lot faster than he swims, at what point should he enter the water?

4. If Elvis runs twice as fast as he swims, at approximately what point should he enter the water?

For Questions 5, think generally about the calculus of any function, not specifically about this problem.

5. Consider an arbitrary, continuous function, T(x), defined on the closed interval [a,b]. Describe how you would attempt to find the value or values of x where T has an absolute minimum.

Part II: Applying Questions 1 – 5 to Specific Cases:

In this section we think about an actual situation where the ball has been thrown to a specific location. This fixes z and y. We will consider how changing r and s can cause the optimum point for Elvis to enter the water to vary.

For each of the following scenarios, let $z = 15.0$ meters and $y = 10.0$ meters.

The time it takes Elvis reach the ball is the sum of the time it takes him on land and the time it takes him in the water.

1. How far does Elvis run on land? Write an expression in terms of x:

2. How much time does Elvis take on land, in terms of x and r?

3. How far does Elvis have to swim? Write an expression in terms of x:

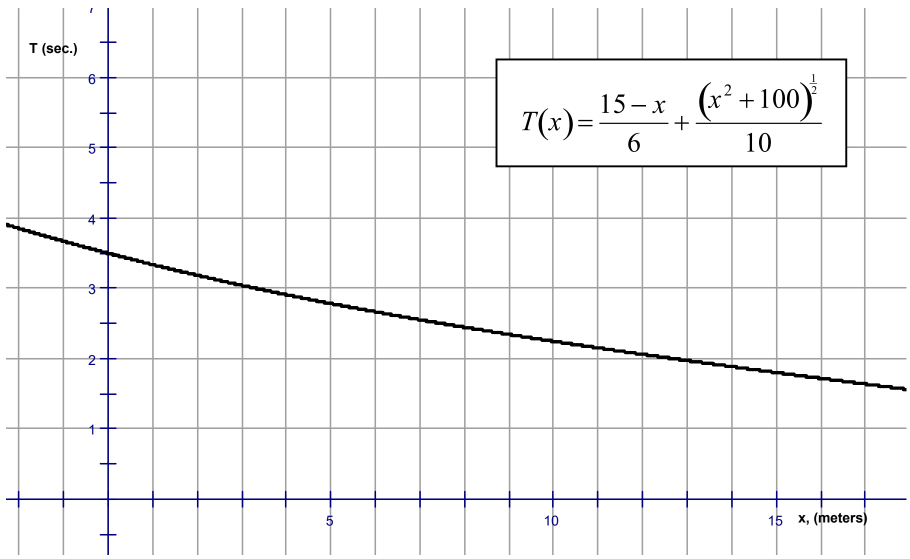
4. How much time does it take Elvis in the water, in terms of x and s?

5. Write the equation of a function, T(x,r,s), that represents the total time it takes Elvis to reach the ball.

For the following four scenarios, we look at fixed values r and s making T a function of x alone.

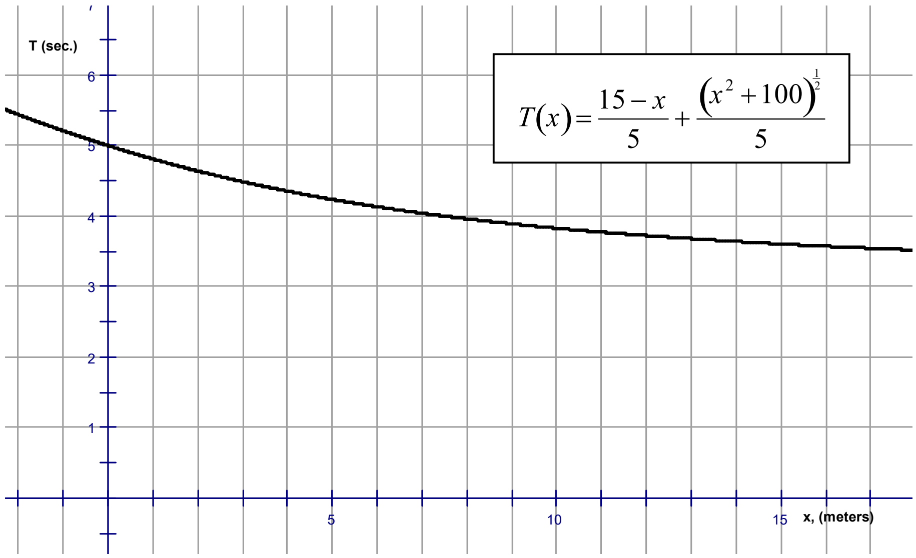
6. What is the domain of T(x)?

The graph of T(x) is shown for the specific case when Elvis' Running Speed is 6.0 meters per second, and his swimming speed is 10.0 meters per second.



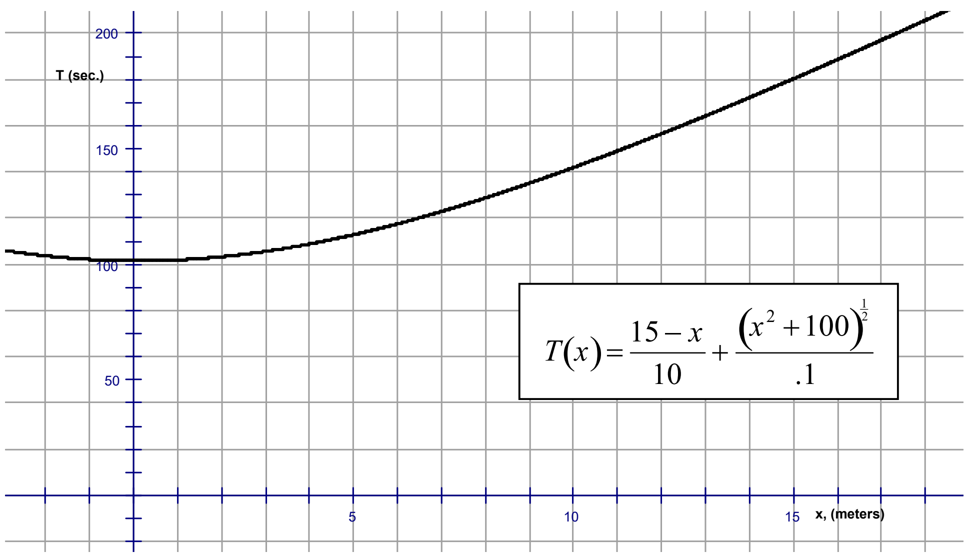
7. Using the graph above, for what value of x is T a minimum? Is this consistent with your physical arguments in part I?

The graph of T(x) is shown for the specific case when Elvis' Running Speed is 5.0 meters per second, and his swimming speed is 5.0 meters per second.



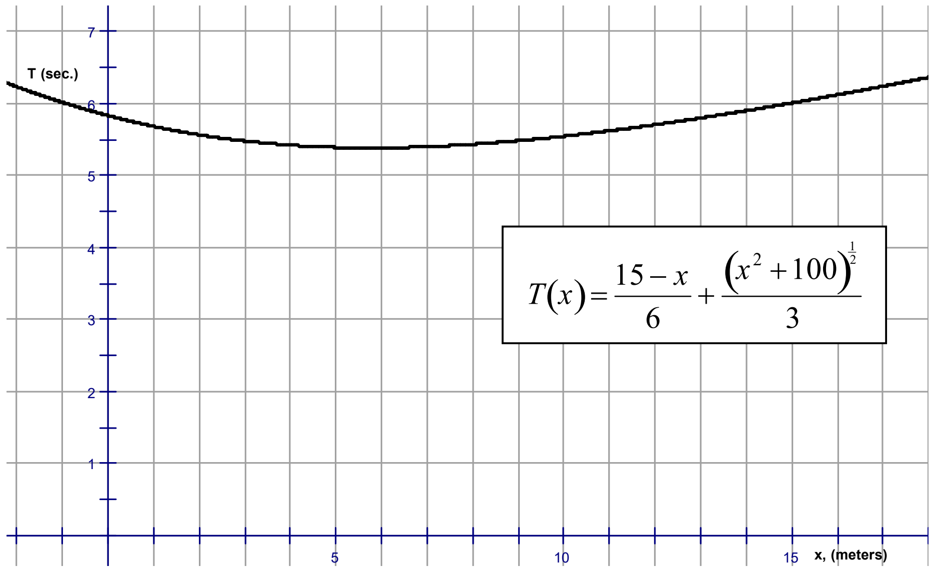
8. Using the graph above, for what value of x is T a minimum? Is this consistent with your physical arguments in part I?

The graph of T(x) is shown for the specific case when Elvis' Running Speed is 10.0 meters per second, and his swimming speed is 0.10 meters per second.



9. Using the graph above, for what value of x is T a minimum? Is this consistent with your physical arguments in part I?

The graph of T(x) is shown for the specific case when Elvis' Running Speed is 6.0 meters per second, and his swimming speed is 3.0 meters per second.



10. Using the graph above, for what value of x is T a minimum? Is this consistent with your physical arguments in part I?

Part III: The General Case

For the following questions, the running and swimming speeds are not given, but they are considered to be constant. z and y are no longer given.

1. Write a general formula for the time it takes to reach the ball, T, in terms of r, s, x, y, and z.

2. For each individual throw, the following quantities are constant:
   The running rate: r
   The swimming rate: s
   The distance along the shore to the ball: z
   The distance in the water to the ball: y

The time to the ball depends only on x; differentiate the function T(x) with respect to x.

3. Solve for x, when T'(x) = 0

4. This x gives us the location of the entry point into the water that minimizes the time to the ball. Using this expression for x, which of the quantities, (r, s, y, z), affect the value of x; which don't?

5. Using this expression for x, we can revisit the analysis we did in part I (qualitatively), and in part II (with function graphs). What does your equation in #3 tell us will happen when:
   a. $r<s$?

   b. $r>>s$?

   c. Is this consistent with the analysis you did in parts I and II; explain.

   d. For a fixed r, what is the smallest value of s for which it makes sense for Elvis to just jump in the water?

6. Using your equation for the minimum value of x and the scenario in the last graph on part II (where r = 6 and s = 3, and y = 10) compute the x-value that minimizes time.

7. Examine the last graph in Section II. Is your answer in question 6 consistent with this graph?

PART IV: ELVIS' DATA

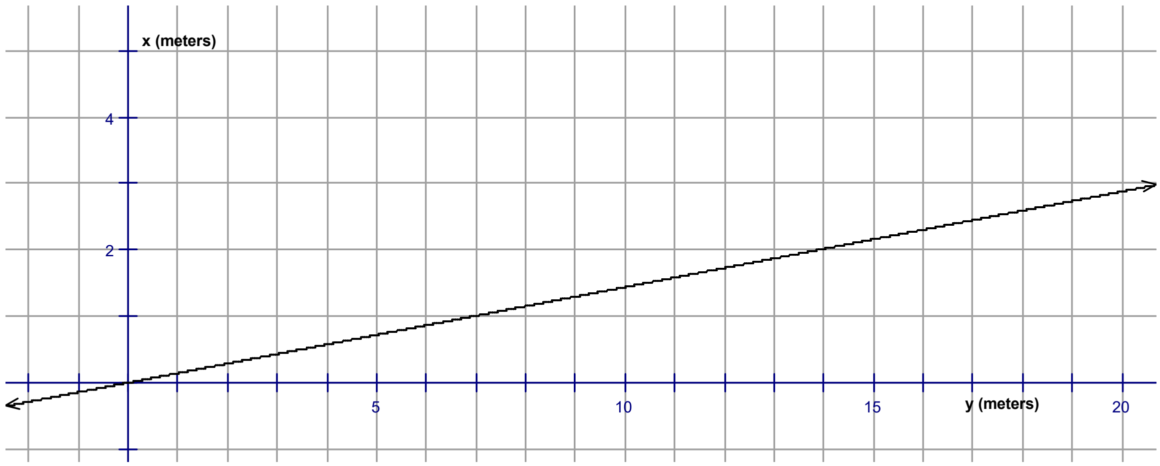
By timing several trials, and averaging Elvis' best times, Tim Pennings computed the average of Elvis' best running and swimming times. He then conducted several ball-throwing trials, measuring x and y each time. With a fixed r and s, the minimum time to reach the ball now depends only on how far from shore the ball is thrown (y), and the location where Elvis enters the water (x).

Using Elvis' average running and swimming times:
$r = 6.40$ meters per second
$s = 0.91$ meters per second

1. Substitute these quantities in the expression for x when $T'(x) = 0$, to determine the relationship between x and y.

2. For this specific situation where r and s are fixed, how does x depend on y?

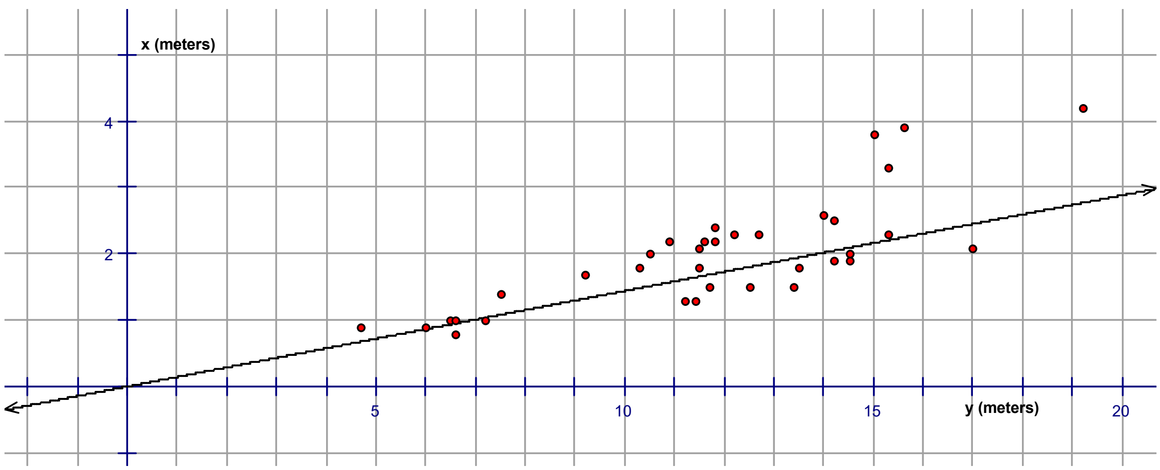
3. The graph of x vs. y is shown on the graph below.


x (meters)
4
2
5
10
15
y (meters)
20

Tim Penning collected the following data, describing where Elvis' entered the water to retrieve the ball for several trials (all distances in meters):

y:	10.5	7.2	10.3	11.7	12.2	19.2	11.4	17.0	15.6	6.6	14.0	13.4
x:	2.0	1.0	1.8	1.5	2.3	4.2	1.3	2.1	3.9	1.0	2.6	1.5
y:	6.5	11.8	4.7	11.6	11.5	9.2	13.5	14.2	14.2	10.9	11.2	15.0
x:	1.0	2.4	0.9	2.2	1.8	1.7	1.8	1.9	2.5	2.2	1.3	3.8
y:	14.5	6.0	14.5	12.5	15.3	11.8	7.5	11.5	12.7	6.6	15.3	
x:	1.9	0.9	2.0	1.5	2.3	2.2	1.4	2.1	2.3	0.8	3.3	

Below is a graph of the data, showing how x depends on y, along with the line from the previous page (representing the optimum x for a given y).



4. Look at the graph above, how does the graph of the data compare with that of the line?

5. Do you feel Elvis does know Calculus; is he jumping in the water at the right value of x so that he minimizes his time to the ball?

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